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ON THE THEORY OF BERGER PLATES*

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It is shown that the Berger equations can be obtained on the basis of a sequential asymptotic procedure. M. Berger /1/ proposed simple approximate nonlinear equations for rectangular and circular plates. The results of /1/ were later extended to orthotropic plates /2/, membranes /3,4/, shallow spherical /4- 8/ and cylindrical /8- 10/ shells. Equations of this type were also used to solve dynamical problems /11- 13/.

The foundation for the Berger equations and the domain of their applicability have repeatedly been discussed in the literature /2,4,5,7- 10,14- 18/. Berger /1/ simply discarded the second invariant of the strain tensor in the expression for the potential energy on the basis that the numerical computations display its slight influence on the state of bending stress. Other authors performed the same operation taking little care about the foundation for similar simplifications, which sometimes resulted in false results. Thus, the erroneousness of the simplified dynamical equations obtained in /12/ for shallow spherical shells is shown in /13/. Hence, it is important to arrive at equations of the Berger type without utilizing the hypothesis about the smallness of the second invariant of the strain tensor.

1. We write the nonlinear equations of motion of a rectangular plate

$$\begin{gathered}
I'_{1,x} + (1-\nu)\{0.5\,\varepsilon'_{12,y} - \varepsilon'_{2,x}\} - \rho(1-\nu^2)\varepsilon^{-1}u^{\cdot\cdot\prime} = 0 \\
I'_{1,y} + (1-\nu)\{0.5\,\varepsilon'_{12,x} - \varepsilon'_{1,y}\} - \rho(1-\nu^2)\varepsilon^{-1}v^{\cdot\cdot\prime} = 0 \\
D(1-\nu^2)\nabla^4 w' - Eh\,[(I_1'w'_{,x})_{,x} + (I_1'w_{,x})_{,x} + \\
(1-\nu)\{(\varepsilon_2'w_{,x}')_{,x} + (\varepsilon_1'w_{,y}')_{,y} - 0.5\,(\varepsilon_{12}'w_{,y}')_{,x} - \\
0.5\,(\varepsilon_{12}'w_{,x}')_{,y}\}] + \rho(1-\nu^2)\,hw^{\cdot\cdot\prime} = 0 \\
I_1' = \varepsilon_1' + \varepsilon_2',\ \varepsilon_1' = u_{,x}' + 0.5\,(w_{,x}')^2,\ \varepsilon_2' = v_{,y}' + 0.5\,(w_{,y}')^2, \\
\varepsilon_{12} = u'_{,y} + v'_{,x} + w'_{,x}w'_{,y} \\
D = \frac{Eh^3}{12(1-\nu^2)},\quad \nabla^4 = \nabla^2\nabla^2,\quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\end{gathered} \tag{1.1}$$

Here I_1' is the first invariant of the strain tensor, E, ρ are the Young's modulus and the density of the plate material, ν is the Poisson's ratio, x, y are orthogonal Cartesian coordinates, u', v' are displacements in the directions of the x, y axes, w' is the normal deflection, the dot denotes differentiation with respect to the time t, and the subscripts x and y differentiation with respect to the corresponding variable.

The appropriate boundary conditions should supplement (1.1). For instance, let

$$\begin{gathered}
\mathbf{U} = (u', v', w') = 0,\ w_{,x}' = 0 \text{ as } x = 0, a; \\
\mathbf{U} = 0,\ w_{,y}' = 0 \text{ as } y = 0, b
\end{gathered} \tag{1.2}$$

The terms in the braces in (1.1) are obtained because of varying the second invariant of the strain tensor.

2. We first turn to the spatially-one-dimensional case /19,20/ and examine the nonlinear vibrations of a rod. The Berger equation then agrees with the known equation obtained on the basis of the Kirchhoff hypothesis (neglecting longitudinal inertia) in /19/. As is shown in /22/, such an equation is obtained as one of the possible limit cases (for sufficiently large variability in the space variable) because of an asymptotic analysis of the initial system. Let us attempt to construct such an asymptotic even in this case by using $\varepsilon = h/2\sqrt{3}a$ as a small parameter. We first perform the following change of variable

$$\xi = \frac{x}{a},\quad \eta = \frac{y}{a},\quad \tau = \sqrt{\frac{\rho(1-\nu^2)}{E}}\,at,\quad \mathbf{U} = \frac{\mathbf{U}'}{a}$$

Equations (1.1) are then rewritten in the following form (the dot now denotes differentiation with respect to τ)

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$$
\begin{aligned}
&I_{1,\xi} + (1-\nu)(0.5\,\varepsilon_{12,\eta} - \varepsilon_{2,\xi}) - u^{\cdot\cdot} = 0 \\
&I_{1,\eta} + (1-\nu)(0.5\,\varepsilon_{12,\xi} - \varepsilon_{1,\eta}) - v^{\cdot\cdot} = 0 \\
&\nabla^4 w - \varepsilon^{-2}\{(I_1 w_{,\xi})_{,\xi} + (I_1 w_{,\eta})_{,\eta} + (1-\nu)[(\varepsilon_2 w_{,\xi})_{,\xi} + (\varepsilon_1 w_{,\eta})_{,\eta} - \\
&\quad 0.5\,(\varepsilon_{12} w_{,\eta})_{,\xi} - 0.5\,(\varepsilon_{12} w_{,\xi})_{,\eta}]\} + \varepsilon^{-2} w^{\cdot\cdot} = 0 \\
&I_1 = \varepsilon_1 + \varepsilon_2,\quad \varepsilon_1 = u_{,\xi} + 0.5\,(w_{,\xi})^2,\ \varepsilon_2 = v_{,\eta} + 0.5\,(w_{,\eta})^2,\quad \varepsilon_{12} = u_{,\eta} + \\
&\quad v_{,\xi} + w_{,\xi} w_{,\eta} \\
&\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}
\end{aligned}
\tag{2.1}
$$

The system (2.1) is nonlinear, hence it is impossible to introduce the concept about variability of the required functions just as simply as in the linear case /21/. Following the ideas in /22,23/, we represent the displacement vector in the form

$$
\mathbf{U} = \mathbf{U}\,(\varepsilon^{\alpha}\theta\,(\xi,\eta),\ \xi,\ \eta,\ \varepsilon) \tag{2.2}
$$

As usual, the exponent α is now selected from the condition of non-contradiction of the appropriate limit systems /21/. In this case one such value is $\alpha = -0.5$. We substitute (2.2) into (2.1) and write the limit system $(\varepsilon \to 0)$. Here we consider the function $\theta\,(\xi,\eta)$ as a new independent variable /22,23/ and take into account that

$$
\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} + \varepsilon^{-1/2}\theta_{,\xi}\frac{\partial}{\partial \theta},\quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} + \varepsilon^{-1/2}\theta_{,\eta}\frac{\partial}{\partial \theta}
$$

We finally have

$$
\begin{aligned}
&I_{1,\theta}\theta_{,\xi} + 0.5\,(1-\nu)\,[u_{,\theta\theta}(\theta_{,\eta})^2 - v_{,\theta\theta}\theta_{,\xi}\theta_{,\eta}] = 0 \\
&I_{1,\theta}\theta_{,\eta} + 0.5\,(1-\nu)\,[v_{,\theta\theta}(\theta_{,\xi})^2 - u_{,\theta\theta}\theta_{,\xi}\theta_{,\eta}] = 0 \\
&w_{,\theta\theta\theta\theta}[(\theta_{,\xi})^2 + (\theta_{,\eta})^2]^2 - \{I_{1,\theta} w_{,\theta}(\theta_{,\xi} + \theta_{,\eta})\} - \\
&\quad I_1 w_{,\theta\theta}[(\theta_{,\xi})^2 + (\theta_{,\eta})^2] + w^{\cdot\cdot} = 0
\end{aligned}
\tag{2.3}
$$

There follows from the first two equations in (2.3)

$$
I_{1,\theta} = 0 \tag{2.4}
$$

Then the term in the braces drops out of the last equation in (2.3). Now, if we return to the variables x, y, t, we then obtain $I_{1,x} = I_{1,y} = 0$ from (2.4). Taking account of the boundary conditions (1.2), we hence determine

$$
I_1 = 0.5\int_0^b\int_0^a [(w_{,x})^2 + (w_{,y})^2]\,dx\,dy \tag{2.5}
$$

Taking (2.4) into account, we obtain the Berger equation for the vibrations of a rectangular plate

$$
h^2\nabla^4 w - 12\nabla^2 w + 12\rho\,(1-\nu^2)\,w^{\cdot\cdot} = 0 \tag{2.6}
$$

from the last equation of (2.3) in the initial variables.

It is possible to arrive analogously at (2.6) by set off from the Karman system of nonlinear equations.

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